Endomorphism RINGS and Baer-Kaplansky classes

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NCRA VII 2021

Bonjour à tous + Thank you to

- the organizers of NCRA 2021 for their hospitality
- ALL of you for your presence

The 2 joint papers in the abstract with

- Derya Keskin Tütüncü (Turkey)
- Rachid Tribak (Morocco)

contain the results in this talk

Theorem 3 & Bad news

This talk will be a VISUAL presentation with many PICTURES

(modules, quivers, Auslander-Reiten quivers)

quiver = oriented graph

AR quiver of an algebra = quiver s.t.

vertices $\longleftrightarrow$ indecomposable modules

arrows $\longleftrightarrow$ "irreducible" maps

RINGS in this talk:

- K field
- K-algebras which are "path algebras" of some quiver
- $\mathbb{Z}$
- endomorphism rings of modules over • • •

Conventions if A = path algebra of Q and Q = quiver with vertices $1, \dots, n$:

- For any $i = 1, \dots, n$ the symbol i denotes also the SIMPLE left A-module (of dimension one) generated by the path of length zero around the vertex i.

- Pictures of the form $\begin{smallmatrix}1\\1\end{smallmatrix}$, $\begin{smallmatrix}1\\2\end{smallmatrix}$, $\begin{smallmatrix} & 1 & \\ 2 & & 3\end{smallmatrix}$, $\begin{smallmatrix}1 & & 2\\ & 3 & \end{smallmatrix}$ denote the <u>indecomposable</u> middle term M of short exact sequences of the form

$$0 \longrightarrow X \longrightarrow M \longrightarrow Y \longrightarrow 0$$

with $X \in \{1, 2, 2 \oplus 3, 3\}$ and $Y = \{1, 1, 1, 1 \oplus 2\}$ respectively.

$\boxed{0 \to 1 \to \begin{smallmatrix}1\\1\end{smallmatrix} \to 1 \to 0}$ contains all the indecomposable modules over the path algebra of $\begin{tikzcd} \underset{1}{\bullet} \arrow[loop right, "a"] \end{tikzcd}$ with relation $a^2 = 0$, isomorphic to $\boxed{K[x]/(x^2)}$, with

AR quiver

$$\begin{tikzcd} & \begin{smallmatrix}1\\1\end{smallmatrix} \arrow[dr] & \\ 1 \arrow[ur] & & 1 \end{tikzcd}$$

$$\boxed{0 \to 2 \to \begin{smallmatrix}1\\2\end{smallmatrix} \to 1 \to 0}$$

contains all the indecomposable modules over the path algebra of the quiver $\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet}$ isomorphic to $\boxed{\begin{pmatrix} K & 0 \\ K & K \end{pmatrix}}$ and with

AR quiver

$$\begin{tikzcd}
& \overset{\begin{smallmatrix}1\\2\end{smallmatrix}}{\bullet} \arrow[dr] & \\
\underset{2}{\bullet} \arrow[ur] & & \underset{1}{\bullet}
\end{tikzcd}$$

$\boxed{0 \to 2 \oplus 3 \to \begin{smallmatrix} 1 \\ 2\;3 \end{smallmatrix} \to 1 \to 0}$ contains 4 indecomposable modules over the path algebra of the quiver

$$\begin{tikzcd} & \overset{1}{\bullet} \arrow[dl] \arrow[dr] & \\ \underset{2}{\bullet} & & \underset{3}{\bullet} \end{tikzcd}$$

isomorphic to

$$\boxed{\begin{pmatrix} K & 0 & 0 \\ K & K & 0 \\ K & 0 & K \end{pmatrix}}$$

and with AR quiver

$$\begin{tikzcd} \overset{2}{\bullet} \arrow[dr] & & \overset{\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}}{\bullet} \arrow[dr] & \\ & \underset{\begin{smallmatrix} 1 \\ 2\;3 \end{smallmatrix}}{\bullet} \arrow[ur] \arrow[dr] & & \bullet\, 1 \\ \overset{3}{\bullet} \arrow[ur] & & \underset{\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}}{\bullet} \arrow[ur] & \end{tikzcd}$$

$$0 \to 3 \to \begin{smallmatrix}1 & & 2\\ & 3 & \end{smallmatrix} \to 1 \oplus 2 \to 0$$

contains 4 indecomposable modules over the path algebra of the quiver

$$\begin{tikzcd}[column sep=small]
\overset{1}{\bullet} \arrow[dr] & & \overset{2}{\bullet} \arrow[dl] \\
& \underset{3}{\bullet} &
\end{tikzcd}$$

isomorphic to $\boxed{\begin{pmatrix} K & 0 & 0 \\ 0 & K & 0 \\ K & K & K \end{pmatrix}}$

with AR quiver

$$\begin{tikzcd}[column sep=small]
& \begin{smallmatrix}1\\3\end{smallmatrix} \arrow[dr] & & 2 \\
3 \arrow[ur] \arrow[dr] & & \begin{smallmatrix}1 & & 2\\ & 3 & \end{smallmatrix} \arrow[ur] \arrow[dr] & \\
& \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[ur] & & 1
\end{tikzcd}$$

Notation

K field, A algebra$/K$, module = left module

$\mathbb{P}$: set of all positive primes

$\mathbb{Z}(p^\infty)$: Prüfer p-group (indecomp. + injective)

J_p : group (or ring) of p-adic integers

GCH = Generalized Continuum Hypothesis

$(\kappa \geq \aleph_0,\ \kappa \leq \alpha \leq 2^\kappa \implies \alpha = \kappa$ or $\alpha = 2^\kappa)$

$\mathcal{C}$ = BAER-KAPLANSKY CLASS :

if X and Y are in $\mathcal{C}$, then

$$\boxed{\mathrm{End}\,X \simeq \mathrm{End}\,Y \implies X \simeq Y}$$

$\simeq$ (rings), $\simeq$ (modules) = isomorphism of rings / modules

Well known Baer-Kaplansky classes:

- all vector spaces over K
- all abelian p groups $(p \in \mathbb{P})$
- all abelian torsion groups

Results on Baer-Kaplansky classes

with <u>1 indecomposable</u> module

(and very "small" or very "large" modules)

<u>Theorem 1</u> If GCH holds, K = field and $\mathcal{C}$ = class of all vector spaces V such that either $\dim_K V < \aleph_0$ or $\dim_K V \geq \max\{\aleph_0, |K|\}$, then for all V_1 and V_2 in $\mathcal{C}$ we have

$$\boxed{\dim_K \operatorname{End} V_1 = \dim_K \operatorname{End} V_2 \implies V_1 \simeq V_2}$$

$$\underline{\text{Theorem 1}} \rightsquigarrow \underline{\text{Corollary A}}$$

$K \rightsquigarrow A$ finite dim. K-algebra

$K \rightsquigarrow {}_AM$ finite dim. BRICK (= module s.t. $\mathrm{End}_A M \simeq K$)

$\mathcal{C} \rightsquigarrow \mathcal{C}$ obviously defined

<u>Corollary A</u>		<u>Corollary B</u>
A	$\rightsquigarrow$	A
${}_A M$ brick	$\longrightarrow$	${}_A M$ finite dim. module s.t. $\mathrm{End}_A M$ = local algebra
....		
C	$\rightsquigarrow$	C obviously defined

Results on Baer-Kaplansky classes with <u>2 indecomposable</u> modules

(formed by finite dimensional modules rarely determined by numerical invariants)

__Theorem 2__ U, V finite dim. A-modules,

$\left.\begin{array}{l}\mathrm{End}_A U \\ \mathrm{End}_A V\end{array}\right\}$ local algebra of dimension $\begin{array}{l}a \\ \underline{b > a}\end{array}$

$\left.\begin{array}{l}\mathrm{Hom}_A(U,V) \\ \mathrm{Hom}_A(V,U)\end{array}\right\}$ vector space of dimension $\begin{array}{l}c \\ d\end{array}$

s.t. $a-c \neq b-d$

$$\Longrightarrow \boxed{\{U^m \oplus V^n \,/\, m, n \in \mathbb{N}\} \text{ Brauer-Koplowsky class}}$$

Theorem 3 U, V, a, b, c, d as in Theorem 2 –

TFAE:

1. $\boxed{X = \bigoplus_{i=1}^{n} X_i, \; Y = \bigoplus_{i=1}^{n} Y_i, \; X_i, Y_i \in \{U, V\}, \; \dim_K \operatorname{End}_A X = \dim_K \operatorname{End}_A Y} \Longrightarrow X \simeq Y$

2. $2a \leq c + d \leq 2b$

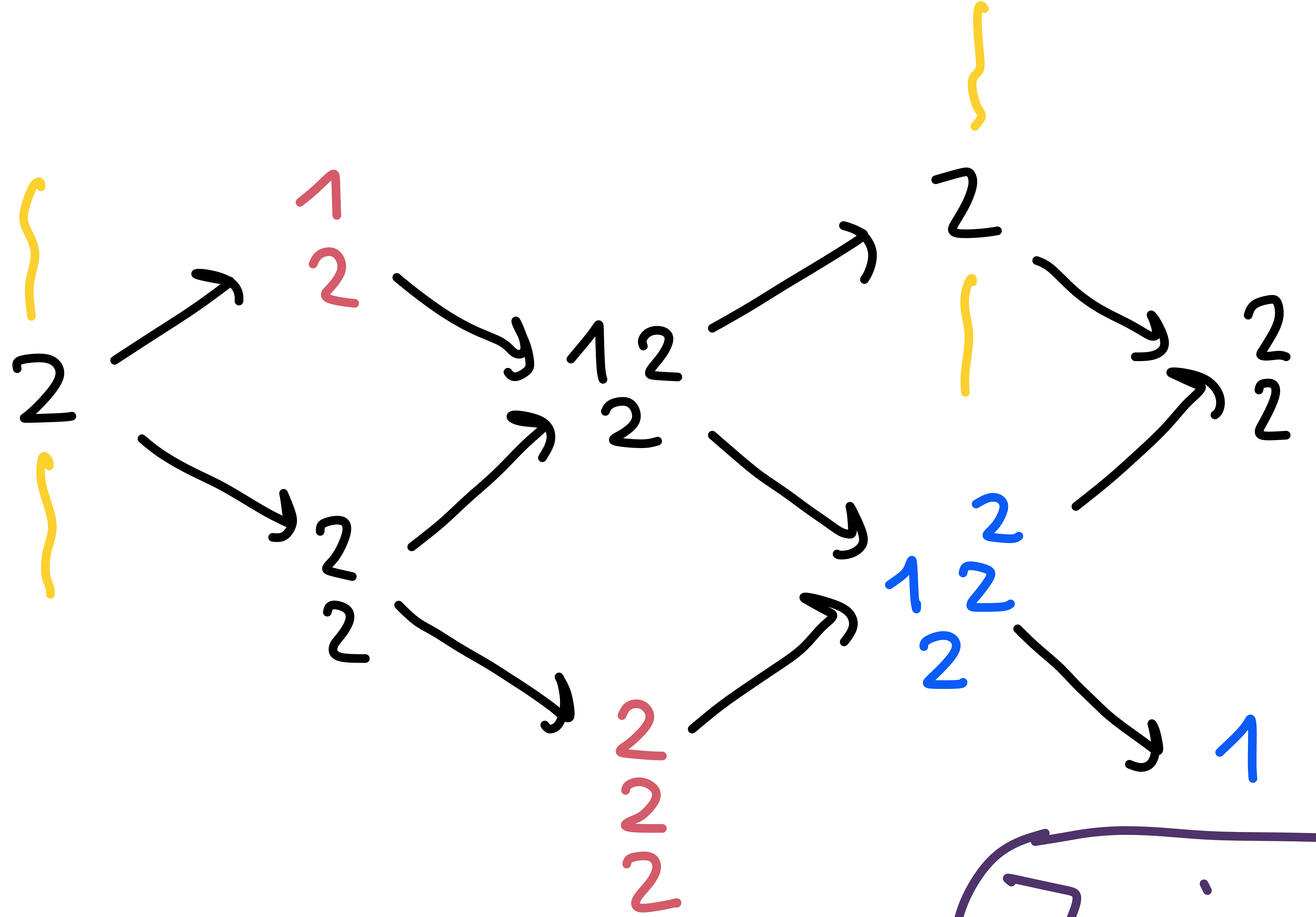
Example satisfying the hypotheses but <u>NOT</u> condition 2 of Theorem 3:

- A path algebra of
$$\begin{tikzcd} 1 \arrow[r, "s"] & 2 \arrow[loop right, "t"] \end{tikzcd}$$
with $ts = 0$, $t^3 = 0$
- $U = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$, $V = \begin{smallmatrix} 2 \\ 2 \\ 2 \end{smallmatrix}$ (or $U = 1$, $V = \begin{array}{cc} & 2 \\ 1 & 2 \\ 2 & \end{array}$)

$$\left.\begin{array}{ll} a = 1, & b = 3 \\ c = 0, & d = 1 \end{array}\right\} \Longrightarrow 2a = 2 > 1 = c + d$$

2 fails

Auslander-Reiten quiver of A



7 indecomposables

Example satisfying the hypotheses and condition (2) of Theorem 3 with $a=1$, $b=2$, $c=1$, $d=1$

$$\left(\Longrightarrow \quad 2a = 2 = 1+1 = c+d < 4 = 2b \right)$$

- A path algebra of $\bullet_1 \underset{t}{\overset{s}{\rightleftarrows}} \bullet_2$ with $st = 0$

$$\boxed{U = \begin{smallmatrix}2\\1\end{smallmatrix},\ V = \begin{smallmatrix}1\\2\\1\end{smallmatrix}} \quad \text{or} \quad \boxed{U = \begin{smallmatrix}1\\2\end{smallmatrix},\ V = \begin{smallmatrix}1\\2\\1\end{smallmatrix}} \quad \text{or} \quad \boxed{U = 1,\ V = \begin{smallmatrix}1\\2\\1\end{smallmatrix}}$$

projective   injective

Auslander-Reiten quiver of A

5 indecomposables

$$
\begin{tikzcd}[column sep=small]
 & & \begin{smallmatrix}1\\2\\1\end{smallmatrix} \arrow[dr] & & & \\
 & \begin{smallmatrix}2\\1\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[dr] & & \\
1 \arrow[ur] & & 2 \arrow[ur] & & 1 &
\end{tikzcd}
$$

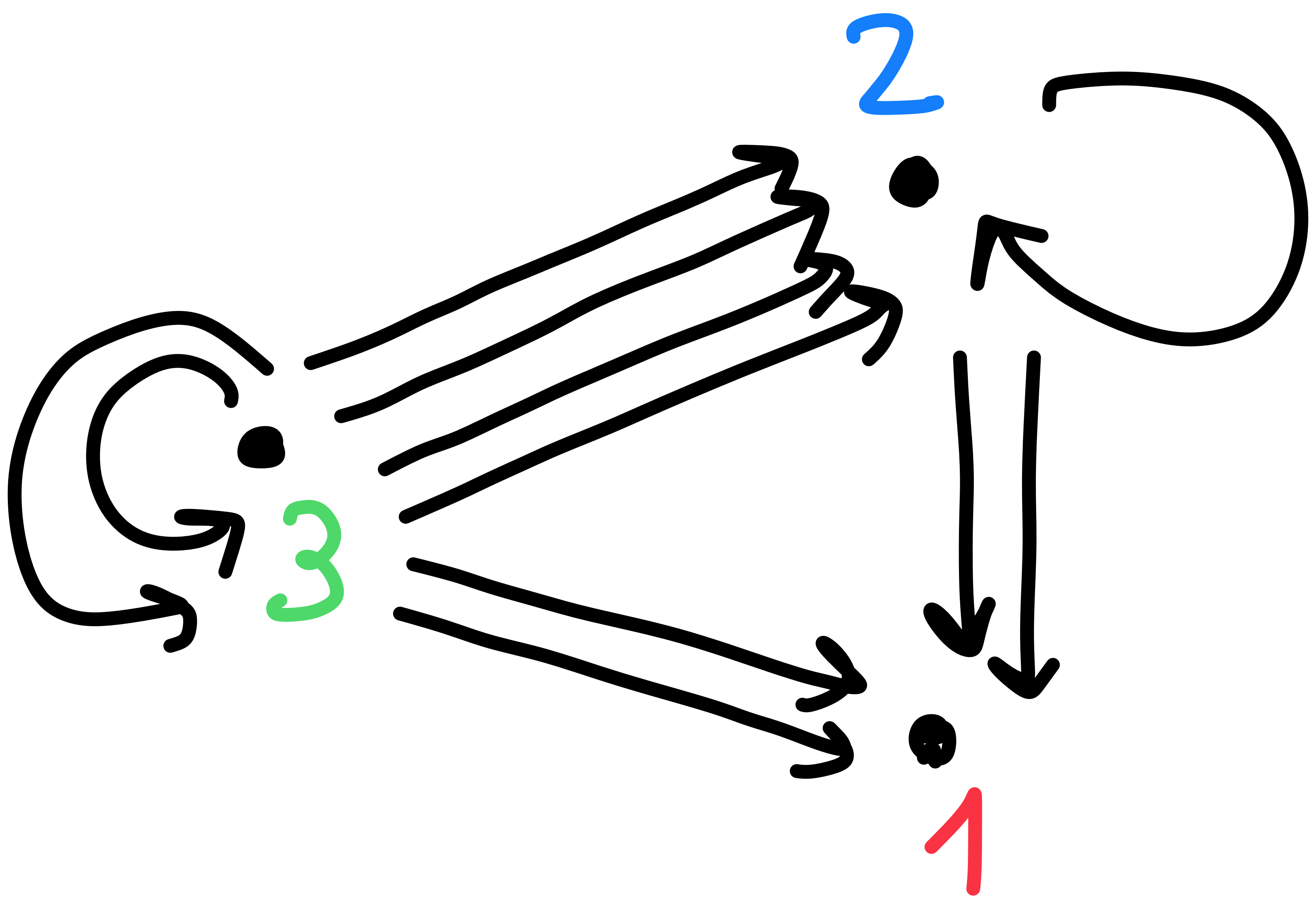
BAD NEWS on classes with

3 indecomposable modules

$\exists\ X_1, X_2, X_3$ such that

- If $i<j$, then $U = X_i$ and $V = X_j$ satisfy the hypotheses and conditions 1 and 2 of Theorem 3
- $\{X_1^{m_1} \oplus X_2^{m_2} \oplus X_3^{m_3} \mid m_i \in \mathbb{N}\}$ contains two direct sums L and M of n indec. modules such that $\boxed{\dim_K \mathrm{End}_A L = \dim_K \mathrm{End}_A M}$ but $L \not\cong M$

Example

- A path algebra of 

- $X_1 = 1$, $X_2 = \begin{array}{c} 2 \\ 1\,1\,2 \end{array}$, $X_3 = \begin{array}{c} 3 \\ 1\,1\,2\,2\,2\,2\,3\,3 \end{array}$

[X_i = projective for $i = 1, 2, 3$]

$\oplus$ of 5 indecomp. modules

$$L = X_2^5$$

$$M = X_1^1 \oplus X_2^1 \oplus X_3^3$$

$$\dim_K \mathrm{End}_A L = 50 = \dim_K \mathrm{End}_A M$$

$\oplus$ of 7 indecomposable modules

$$L = X_1^{2} \oplus X_3^{5}$$

$$M = X_2^{6} \oplus X_3^{1}$$

$$\dim_K \operatorname{End}_A L = 99 = \dim_K \operatorname{End}_A M$$

2 classes of ABELIAN GROUPS

$\mathcal{C}$ with <u>∞ many</u> indecomposables

s.t. for all G and H in $\mathcal{C}$

$$\boxed{(\mathrm{End}\, G, +) \simeq (\mathrm{End}\, H, +) \Longrightarrow G \simeq H}$$

G abelian group

T = torsion subgroup of G

$$T = \bigoplus_{p \in \mathbb{P}} T_p$$

$$T_p = \{x \in T \mid p^n x = 0 \text{ for some } n \geq 1\}$$

$G = D \oplus R$ with

D divisible $(x \in D, n \geq 1 \Rightarrow \exists y \in D : ny = x)$

R reduced (without nou zero divisible summands)

Theorem 4 If GCH holds and $\mathcal{C}$ = class of all DIVISIBLE ABELIAN GROUPS $D = V \oplus T$ s.t.

- $|\mathrm{soc}\, T_p| < \aleph_0$ for any $p \in \mathbb{P}$
- $V \neq 0 \Longrightarrow T_p = 0$ for almost all $p \in \mathbb{P}$
- $T \neq 0 \Longrightarrow \dim_{\mathbb{Q}} V \geq \aleph_0$

then $\boxed{(\mathrm{End}\, G, +) \simeq (\mathrm{End}\, H, +) \Longrightarrow G \simeq H}$

Warning We cannot delete $\bullet$

Example $G = \mathbb{Q} \oplus \mathbb{Z}(p^\infty)$

$H = \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Z}(p^\infty)$

$$\boxed{(\mathrm{End}\, G, +) \cong W \oplus J_p \cong (\mathrm{End}\, H, +),\ \dim_{\mathbb{Q}} W = 2^{\aleph_0}}$$

but $G \not\cong H$

Theorem 5 If GCH holds and $\mathcal{C}$ = class of all TORSION FREE ABELIAN GROUPS $G = V \oplus R$, V divisible, R reduced s.t.

- $R = 0$ or $R = \bigoplus_{p \in S} J_p$, $S \subseteq \mathbb{P}$
- $V \neq 0$ and $R \neq 0 \Rightarrow \dim_{\mathbb{Q}} V \geq 2^{\aleph_0}$

then $\boxed{(\mathrm{End}\, G, +) \simeq (\mathrm{End}\, H, +) \Longrightarrow G \simeq H}$

<u>Warning</u> We cannot delete • or replace it with the hypothesis that $\dim_{\mathbb{Q}} V \geq \aleph_0$

<u>Example</u>

$$G = V_1 \oplus J_p \quad , \dim_{\mathbb{Q}} V_1 = \aleph_0$$

$$H = V_2 \oplus J_p \quad , \dim_{\mathbb{Q}} V_2 = 2^{\aleph_0}$$

$$\boxed{(\mathrm{End}\, G, +) \simeq W \oplus J_p \simeq (\mathrm{End}\, H, +) \text{ with } \dim_{\mathbb{Q}} W = 2^{2^{\aleph_0}}}$$

but $G \not\cong H$

THANK YOU FOR YOUR ATTENTION !

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